Notes on Topology

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Abstract

It is presented an introduction to the basic concepts in topology. Emphasis is put on homotopy theory. A topological invariant of compact manifolds, the winding number, is defined.

Contents

1   Definitions

A set X together with a set $\mathcal{T} = \{U_i | i \in I\}$ of subsets U_i of X ($\mathcal{T} \ni U_i \subset X = \cup_{i \in I} U_i$) is a **topological space** iff $\mathcal{T}$ contains: (i) the null set and X ($\emptyset, X \in \mathcal{T}$), ($ii$) the union of any of its subsets ($\cup_{j \in J} U_j \in \mathcal{T}$, for any subcollection J of I), (iii) the intersection of every of its *finite* subsystems ($\cap_{k \in K} U_k \in \mathcal{T}$, for every finite subcollection K of I).

The sets $U_i \in \mathcal{T}$ are called **open** sets. A set $A \subset X$ whose complement in X is open is called a **closed** set, *i.e.* (X
A) $\in \mathcal{T}$.

A map $f : X \to Y$ between two topological spaces X and Y is called **continuous** iff the inverse image of every open set in Y is an open set in X, *i.e.* for every $U \in \mathcal{T}_Y$, $f^{-1}(U) \in \mathcal{T}_X$.

A **homeomorphism** is a bijection which is bicontinuous (f is continuous and has an inverse f^{-1} which is also continuous). Two topological spaces are said *homeomorphic* if there is an homeomorphism between them.

map $f : X \to Y$ between two topological spaces X and Y

A **neighborhood** N of a point x in X is a set N_x containing an open set which contains the point x. Then, a subset $A \subset X$ is open iff it contains a neighborhood of each of its points.

A topological space is called a **Hausdorff** space iff any two points in X have disjoint neighborhoods.

A system $\{A_i\}$ of [open] subsets of X is a [open] **covering** of X iff $\cup A_i = X$.

A topological space X is **connected** iff it cannot be written as the union $X = X_1 + X_2$ of two open disjoint $X_1 \cap X_2 = \emptyset$ sets.

A topological space X is **arcwise connected** iff given two points in X there is a continuous path between them. X is **locally arcwise connected** if for each point $x \in X$ and each neighborhood V_x there is a neighborhood $U_x \subset V_x$ which is arcwise connected.

A loop in a topological space X is a continuous map $f : [0, 1] \to X$ such that $f(0) = f(1)$. A topological space X is called **simply connected** iff any loop in X can be continuously shrunk to a point. [1]

2   Compactification

X is **compact** iff every open covering of X has a finite subcovering. A set $A \subset X$ is **compact** iff it is **Hausdorff** and every covering of A has a finite subcovering. It is **locally compact** iff every point has a compact neighborhood.

Consider a locally compact Hausdorff space X, and let $X^+ \equiv X \cup \{z\}$ where z is not an element of X. Let $\mathcal{T}_{X+}$ contain $\mathcal{T}_X$, the complement in X^+ of all compact subsets of X, and X^+. X^+ together with the topology $\mathcal{T}_{X+}$ is a

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[1] the definition is made precise later in terms of fundamental homotopy groups

compact topological space, X being a topological subspace of X^+. X^+ is called the **one-point compactification** of X.

E.g., the one-point compactification of the non-compact space R^n to a compact space $S^n = R^n \cup \{\infty\}$ may be achieved by mapping all points at infinity to a point.

This procedure will be interesting as sometimes it is assumed that the field approach some asymptotic form at spatial infinities, corresponding to the vacuum, or for that matter one of the vacua

3 Homotopy

Homotopy describes continuous deformations of maps one to another, between two topological spaces X and Y. X is chosen as some standard topological space whose structure is well known, as the $n-$sphere, S^n. Homotopy groups then describe how maps from S^n to Y are classified according to homotopic equivalence.

3.1 Definition

A **path** in a topological space X is a continuous map $\alpha : [0,1] \to X$. A **loop** at $x \in X$ is a path α in X such that $\alpha(0) = \alpha(1) = x$. A loop $c_x : [0,1] \to X$ such that $s \mapsto x$ is called a **constant path**.

The **product** $\alpha * \beta$ of two paths α and β in X is defined as

$$\begin{aligned} \alpha * \beta(s) \quad &= \quad \alpha(2s) \qquad & 0 \le s \le \tfrac{1}{2} \\ &= \quad \beta(2s - 1) \qquad & \tfrac{1}{2} \le s \le 1 \end{aligned}$$

The **inverse path** α^{-1} of α is the defined by $\alpha^{-1}(s) \equiv \alpha(1 - s)$.

Two maps $\alpha, \beta : X \to Y$ are said to be **homotopic** $\alpha \sim \beta$ if there exists a continuous mapping $F : X \times [0,1] \to Y$ such that $F(s,0) = \alpha(s), F(s,1) = \beta(s)$. Then F is called a **homotopy** between α and β.

If X is arcwise connected and locally arcwise connected, it is simply connected if every loop is homotopic to a constant map.

Homotopy between loops at a given base point defines an *equivalence relation* $' \sim '$. [2] The equivalence class of loops $[\alpha]$ to which α belongs is called the **homotopy class** of α.

3.2 Fundamental groups

The **fundamental group** or **first homotopy group** $\pi_1(X, x_0)$ of a topological space X at x_0 is the set of homotopy classes of loops at $x_0 \in X$. The group

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[2]*I.e.*, it satisfies: reflexivity, $\alpha \sim \alpha$; symmetry, $\alpha \sim \beta \Rightarrow \beta \sim \alpha$; transitivity, $\alpha \sim \beta \wedge \beta \sim \gamma \Rightarrow \alpha \sim \gamma$.

structure is given by the product of homotopy classes defined as $[\alpha]*[\beta] \equiv [\alpha*\beta]$, using the product of loops defined above.

In case X is an arcwise connected topological space, $\pi_1(X, x_0)$ is isomorphic to $\pi_1(X, x_1)$; therefore the base point may be omitted and can refer simply to $\pi_1(X)$.

An arcwise connected space is simply connected if it has a trivial fundamental group, $\pi_1(X) = \{e\}$.

For arcwise connected topological spaces X and Y, there is the following group isomorphism

$$\pi_1(X \times Y, (x_0, y_0)) \;\cong\; \pi_1(X, x_0)\pi_1 \;\oplus\; (Y, y_0)$$

A fundamental group is invariant under homeomorphisms and hence is a topological invariant.

3.3   Higher homotopy groups

Besides the homotopy classes of *loops* in a topological space X, describing the fundamental group, other groups may be assigned to X, e.g. by considering homotopy classes of *spheres* or *tori*. The focus here is on the group of the homotopy classes of the sphere S^n.

Let I^n be the unit n-cube, $I^n \equiv \{ \ (s_1, \ldots, s_n) \mid 0 \leq s_i \leq 1 \ (1 \leq i \leq n) \ \}$ and denote its boundary by $\partial I^n \equiv \{ \ (s_1, \ldots, s_n) \in I^n \mid some \ s_i \ = \ 0 \ or \ 1 \ \}$.

A n-**loop** at $x_0 \in X$ is a continuous map

$$\alpha : I^n \to X$$

which maps the boundary ∂I^n to the point x_0. Shrinking the boundary if I^n to a point forms the sphere S^n, $I^n/\partial I^n \sim S^n$.

Two n-loops $\alpha, \beta : I^n \to X$ at $x_0 \in X$ are said to be **homotopic** $\alpha \sim \beta$ if there exists a continuous mapping $F : I^n \to X$ such that $F(s_1, \ldots, s_n, 0) = \alpha(s_1, \ldots, s_n)$, $F(s_1, \ldots, s_n, 1) = \beta(s_1, \ldots, s_n)$, $F(s_1, \ldots, s_n, t) = x_0$ for $(s_1, \ldots, s_n) \in \partial I^n$, $t \in I$. F is called a **homotopy** between α and β.

Homotopy between n-loops at a given base point defines an *equivalence relation* $' \sim '$. The equivalence class of n-loops $[\alpha]$ to which α belongs is called the **homotopy class** of α.

The **product** $\alpha * \beta$ of two n-loops α and β in X is defined as

$$
\begin{aligned}
\alpha * \beta(s_1, s_2, \ldots, s_n) \;&=\; \alpha(2s_1, s_2, \ldots, s_n) && 0 \leq s_1 \leq \tfrac{1}{2} \\
&=\; \beta(2s_1 - 1, s_2, \ldots, s_n) && \tfrac{1}{2} \leq s \leq 1
\end{aligned}
$$

The inverse α^{-1} of α is the defined by $\alpha^{-1}(s_1, s_2, \ldots, s_n) \equiv \alpha(1-s_1, s_2, \ldots, s_n)$.

The n**th homotopy group** $(n \geq 1)$ $\pi_n(X, x_0)$ of a topological space X at x_0 is the set of homotopy classes of n-loops at $x_0 \in X$. The group structure is

given by the product of homotopy classes defined as $[\alpha] * [\beta] \equiv [\alpha * \beta]$, induced by the product of n-loops defined above.

In case X is an arcwise connected topological space, $\pi_n(X, x_0)$ is isomorphic to $\pi_n(X, x_1)$; therefore the base point may be omited and can refer simply to higher homotopy group as $\pi_n(X)$.

Higher homotopy groups ($n > 1$) are always **Abelian**,

$$[\alpha] * [\beta] = [\beta] * [\alpha] \qquad (n \geq 2)$$

For arcwise connected topological spaces X and Y, there is the following group isomorphism

$$\pi_n(X \times Y) \;\cong\; \pi_1(X) \;\oplus\; \pi_1(Y)$$

Higher homotopy groups are invariant under homeomorphisms and hence are topological invariants.

For the case $n = 0$, $\pi_0(X)$ is the 0th *homology group* and denotes the number of (arcwise) connected components of X.

3.4   Examples

Lie groups

$$
\begin{aligned}
\pi_1(G) \quad &\cong \quad Z && G = U(1) \\
& \quad Z_2 && G = SO(n) \ (n \geq 3) \\
& \quad \{e\} && other \ simple \ compact \ connected \ Lie \ group
\end{aligned}
$$

$$
\pi_2(G) \quad \cong \quad \{e\} \qquad\qquad G \ any \ compact \ connected \ Lie \ group
$$

$$
\pi_3(G) \quad \cong \quad Z \qquad\qquad G \ any \ simple \ compact \ connected \ Lie \ group
$$

$$
\begin{aligned}
\pi_4(G) \quad &\cong \quad Z_2 \times Z_2 && G = SO(4), \ Spin(4) \\
& \quad Z_2 && SU(2), \ SO(3), \ Spin(5), \ SO(5) \\
& \quad \{e\} && G = SU(n) \ (n \geq 3), \ SO(n) \ (n \geq 6), \ G_2, \ F_4, E_n
\end{aligned}
$$

$$
\pi_{2n}(SU(n)) \quad \cong \quad Z_{n!}
$$

Spheres

$$\begin{aligned}
\pi_n(S^m) &\cong {e} & for\ n < m \\
\pi_n(S^n) &\cong Z \\
\pi_{n+1}(S^n) &\cong Z_2 & except & \quad \pi_2(S^1) \cong {e},\ \pi_3(S^2) \cong Z \\
\pi_{n+2}(S^n) &\cong Z_2 & except & \quad \pi_3(S^1) \cong {e} \\
\pi_{n+3}(S^n) &\cong Z_{24} & except & \quad \pi_4(S^1) \cong {e},\ \pi_5(S^2) \cong Z_2, \\
& & & \quad \pi_6(S^3) \cong Z_{12},\ \pi_7(S^4) \cong Z \times Z_{12} \\
\pi_n(S^1) &\cong {e} & except & \quad \pi_1(S^1) \cong Z
\end{aligned}$$

Torus

$$T^2 = S^1 \times S^1 \qquad\qquad\qquad (torus)$$
$$\pi_1(T^2) \cong \qquad\qquad \pi_1(S^1) \oplus \pi_1(S^1) \cong Z \oplus Z$$

$$T^n = S^1 \times S^1 \times \ldots \times S^1 \qquad\qquad (n - torus)$$
$$\pi_1(T^n) \cong \qquad \pi_1(S^1) \oplus \pi_1(S^1) \oplus \ldots \oplus \pi_1(S^1) \cong Z \oplus Z \oplus \ldots \oplus Z$$

Other relations

$$\pi_n(RP^n) \cong \pi_n(S^n) \cong Z \qquad n \geq 2$$

$$\pi_1(SO(N)) \cong \pi_{n+1}(S^n):$$
$$\pi_1(SO(2)) \cong \pi_3(S^2) \cong Z$$
$$\pi_1(SO(3)) \cong \pi_4(S^3) \cong \pi_4(SU(2)) \cong \pi_4(SO(3)) \cong Z_2$$

Bott periodicity theorem

For $n \geq \frac{(k+1)}{2}$, $k \geq 2$,

$$\begin{aligned}
\pi_k(U(n)) \cong \pi_k(SU(n)) &\cong {e} & k\ even \\
&\cong Z & k\ odd
\end{aligned}$$

For $n \geq (k + 2)$, $k \geq 2$,

$$\begin{aligned}
\pi_k(SO(n)) \cong\ & Z & k = 3, 7\ (mod\ 8) \\
& Z_2 & k = 0, 1\ (mod\ 8) \\
& {e} & k = 2, 4, 5, 6\ (mod\ 8)
\end{aligned}$$

Coset spaces

For any Lie group G and any Lie subgroup $H \subset G$, $\pi_2(G/H)$ is the subgroup of $\pi_1(H)$ that maps into the trivial element of $\pi_1(G)$ when H is embedded in G,

$$\pi_2(G/H) = ker\{\pi_1(H) \mapsto \pi_1(G)\}$$

A special case is

$$\pi_2(G/H) = \pi_1(H) \quad for \quad \pi_1(G) = 0$$

4 Topological invariants

A topological invariant is a property of topological spaces which is preserved under a homeomorphism. Naturally then, if two topological spaces have different topological invariants they cannot be homeomorphic to each other.

A topological invariant may be the number of connected components of a space, connectedness, compactness, separation properties, such as the Hausdorff property.

very useful and well-known topological example is the **Euler characteristic**. Let $X \subset R^3$, and K *any polyhedron* homeomorphic to X; the Euler characteristic $\chi(X)$ of X is given by

$$\chi(X) = \quad (number\ of\ vertices\ in\ k) \ - \ (number\ of\ edges\ in\ k)$$
$$+ \ (number\ of\ faces\ in\ k)$$

This is a good definition as is ensured by the *Poincaré-Alexander* theorem that $\chi(X)$ is indeed independent of the polyhedron K homeomorphic to X considered.

The **homology** groups [3] are refinements in a sense of the Euler characteristic concept.

Homeomorphism define an equivalence relation. Another equivalence relation, which is coarser and quite useful, is that *of the same homotopy type*, as was seen previously (the condition that the continuous functions need to have inverses is relaxed). To see the destinction note that for example the open interval $(0, 1)$ and a point $\{0\}$ are of the same homotopy type although not homeomorphic.

5 The Cartan-Maurer integral invariant

Here it is introduced a topological invariant of compact manifolds which can be written as an integral over the manifold.

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[3] these are properly defined by introducing concepts such as simplexes, simplicial complexes, triangulation of the space, chain and boundary groups, ...; also it is shown, via *de Rham's theorem*, to be the dual of the cohomology group

Consider a mapping $g : \mathcal{S} \rightarrow \mathcal{M}$ from a compact manifold $\mathcal{S}$ of odd [4] dimensionality d into a manifold $\mathcal{M}$ of matrices, with $det\ g\ \neq 0$. In particular, take the $d-$sphere $\mathcal{S} \equiv S^d$ and the representation of some Lie group $\mathcal{M} \equiv G$, respectively.

The **Cartan-Maurer** invariant is given by

$$\mathcal{F}([g]) \equiv \int_{\mathcal{S}} d^d\theta\ \epsilon^{i_1 i_2 \dots i_d}\ tr\ \{g^{-1}(\theta)\ \partial_{i_1}g(\theta)\ \ g^{-1}(\theta)\ \partial_{i_2}g(\theta)\ \dots\ g^{-1}(\theta)\ \partial_{i_d}g(\theta)\}$$

The integral in $\mathcal{F}$ is independent of the local coordinates used to parameterize the manifold $\mathcal{S}$. [5]

Furthermore, it is shown that $\mathcal{F}$ is invariant under *small* deformations of the map g, and therefore it is indeed a function $\mathcal{F}([g])$ of the homotopy class $[g]$ of g. Actually, $\exp \mathcal{F}([g])$ provides a representation of the homotopy group $\pi_d(\mathcal{M})$, as

$$\mathcal{F}([g] * [f])\ =\ \mathcal{F}([g])\ +\ \mathcal{F}([f])$$

In particular,

$$\mathcal{F}([g]^n)\ =\ n\ \mathcal{F}([g])$$

Thus, in case $\mathcal{F}([f]) \neq 0$ for some $[f]$, then $\{[f]^n\}$ form a subgroup Z of $\pi_d(\mathcal{M})$.

A simplest example is provided by $\pi_1(S^1)$, where a representative of the nth class is given by the mapping $g_n(\theta) \equiv \exp in\theta\ (0 \leq \theta \leq 2\pi)$, for which

$$\mathcal{F}([g_n]) = \int_0^{2\pi}\ d\theta\ \exp(-in\theta)\ \partial_\theta \exp(in\theta) = i2\pi n$$

thus confirming $\pi_1(S^1) = Z$.

Consider the case $d = 3$. It is shown that for any simple Lie group G, all continuous mappings $S^3 \rightarrow G$ may be continuously deformed into mappings of S^3 into a *standard* $SU(2)$ subgroup of G. In the case of $G = SU(n)$, this standard subgroup is the one that acts on the first two components of its defining representation.

6   Winding number

For the *identity* map $g_1 : S^3 \rightarrow SU(2)$, from S^3 to the $SU(2)$ subgroup of G, and taking the standard representation and usual generators for the standard $\mathcal{L}(SU(2))$ subalgebra, it turns out

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[4] were it even, the cyclic property of the trace would render the correspondent definition below trivial

[5] which follows from the transformation properties of the ϵ tensor, essentially the Jacobian of the performed change of coordinates

$$\mathcal{F}([g_1]) = 24\pi^2$$

and therefore

$$\mathcal{F}([g_1]^n) = 24\pi^2 n$$

The integer n, or

$$W[g] := \frac{1}{24\pi^2}\mathcal{F}([g_1]^n)$$

is called the **winding number**.

Accordingly, for every simple Lie group G, $\pi_3(G)$ contains Z. Actually, $\pi_3(G) = Z$ for all simple Lie groups; *i.e.*, the homotopy class $[g]$ for any simple Lie group is entirely determined by the homotopy class when the group is deformed into its standard $SU(2)$ subgroup.